Axial segregation in a cylindrical centrifuge

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We propose a theory for axial segregation of suspensions of non-neutrally buoyant particles in a rotating cylinder. We show that the hydrodynamic interaction between pairs of particles produces a relative motion in the axial direction, independent of the gravitational force. If the particles are denser than the suspending fluid, differential centrifuging between particles at different radial positions leads to an attractive interaction, inducing a rapid growth of axial density perturbations. We suggest that this mechanism can explain the origin of band formation in rotating suspensions of non-neutrally buoyant particles.

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A homogeneous suspension of non-neutrally buoyant particles in a rotating cylinder can be unstable to axial perturbations in concentration. In a recent experiment [1], dilute suspensions of non-Brownian silica particles, radius $a = 100 \mu m$ and specific gravity $\rho_s/\rho_f \geq 2$, were suspended in glycerol-water mixtures up to 100 times as viscous as water. After a few hundred rotations of the cylinder, typically at an angular velocity of $\Omega \sim 1$ rad/sec, density fluctuations along the symmetry axis were observed, eventually leading to well defined bands of high and low particle concentration. Similar experiments with neutrally buoyant particles do not show any axial segregation unless the cylinder is only partially filled [2]. A mechanism for axial segregation in the presence of a free surface was recently proposed, based on shear-induced diffusion of the particles [3]. In this paper, we develop a theory for a filled cylinder, which predicts segregation when the particles are denser than the solvent. The theory has no adjustable parameters, yet the calculated growth rate of the most unstable mode is consistent with the band onset times observed experimentally.

The theory is limited to dilute suspensions, and thus the particle trajectories were derived from the dynamics of an isolated particle in a rotational flow. The hydrodynamic interaction between a pair of particles was then calculated by the method of reflections, taking only the leading-order terms. We found a small relative velocity between the particles, which is always attractive, and which eventually leads to axial segregation. However, to a first approximation, the effect of hydrodynamic interactions on the particle trajectories can be ignored in comparison with the much larger rotational flow. This assumption is reasonable if the angular velocity of the cylinder is much greater than the rate of axial segregation, which turns out to be true under the conditions of the experimental observations.

An isolated particle located on a cross section of the rotating cylinder at $r = (r, \theta)$, experiences a gravitational force $-m_B g \hat{j}$ and a centrifugal force $m_B \Omega^2 r \hat{\Omega}$ (Fig. 1(a)), where $m_B = 4/3 \pi a^3 (\rho_s - \rho_f)$ is the buoyancy corrected mass of the particle. In isolation these forces give rise to a settling velocity $-u_s \hat{j}$, where $u_s = m_B g / 6 \pi \mu a$, and a centrifuging velocity $u_c (r/R) \hat{\Omega}$, where $u_c = m_B \Omega^2 R / 6 \pi \mu a$. For the range of experimental parameters where band formation is observed (viscosity $\mu > 10$ cp and angular velocity $\Omega < 3$ rad/s), the Stokes number for settling ($St_s = \rho_s u_s / \mu < 10^{-1}$) and for centrifuging ($St_c = \rho_s u_c a / \mu < 10^{-3}$) are both small. Inertia can therefore be neglected in these calculations and the particle velocity determined from the instantaneous force balance. The velocity $\mathbf{u}$ of an isolated particle located at $r$ is then,

$$u_r = u_s \left( \frac{r}{D_1 R} - \sin \theta \right), \quad u_\theta = u_s \left( \frac{r}{D_2 R} - \cos \theta \right), \quad (1)$$

where $D_1 = g / \Omega^2 R = u_s / u_c$ and $D_2 = u_s / \Omega R$ are two independent dimensionless numbers describing the relative magnitudes of the three forces (gravitational, centrifugal, and Stokes drag) acting on the particle. The effect of the cylinder wall on the particle dynamics is neglected here, although we subsequently consider the screening of the hydrodynamic interactions by the cylinder wall. Equating $u_r$ and $u_\theta$ to zero yields two circles (centers $C_1$ and $C_2$) with diameters of $D_1 R$ and $D_2 R$, on which velocity components vanish (Fig. 1(b)). Usually a particle follows the local stream velocity, but it goes against the stream inside the circle $C_2$, where the downward gravitational force is larger than the upward drag force. Similarly, a particle usually migrates toward the cylinder wall, but moves towards the central axis inside the circle $C_1$, where the downward gravitational force is larger than the upward centrifugal force. The intersection of these circles $(P)$ is a point of unstable equilibrium, where all three forces balance, and is the dynamical center of this system. A particle initially placed near this point will spiral outwards, eventually ending up with a limiting closed trajectory determined by the cylinder wall (Fig. 1(c)). A qualitative prediction of the limiting trajectories of a single particle can be made based on the size of these circles within a finite radius cylinder. For example, if $D_2 > 1$ a particle cannot have a closed trajectory; instead it moves...
As the particle rotates in a nearly closed trajectory, it is the centrifugal velocity component in addition to the gravitational velocity component that leads to band formation. The centrifugal velocity, $u_c$, is proportional to the radial position of the particle, $r$, and it approaches the limiting closed trajectory as $r \to 0$. Thus for the purpose of determining the onset of the axial instability, it will be assumed that the suspension is homogeneous in the transversal plane.

Under the experimental conditions where band formation is observed, $D_1 \sim 10^3$ and $D_2 \sim 10^{-2}$. Because the radius of $C_1$ is so large, while that of $C_2$ is so small, the unstable equilibrium point, $P$, is almost exactly on the $i$-axis, displaced from the center of the cylinder by $D_2 R$. The velocity of a particle relative to the stream, $\vec{v}$, can be expressed as

$$\vec{v} = \frac{\vec{u} - \vec{r} \hat{\theta}}{\Omega R} = D_2 \left( \vec{r} - \frac{\vec{r}}{D_1 R} \right).$$

This expression represents the weak centrifugal velocity, $u_c$, and the particle eventually approaches the limiting closed trajectory, but with a very long time scale $\sim D_1 / D_2 \Omega$. Thus for the purpose of determining the onset of the axial instability, it will be assumed that the suspension is homogeneous in the transversal plane.

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cance in the axial direction, where there is otherwise no relative motion.

These conclusions are quantitatively though not qualitatively affected by the presence of the cylinder wall. The lubrication force between a particle and the cylinder wall reduces the radial velocity when the particle comes in close proximity to the wall [5]. This in turn reduces the differential centrifuging driving the axial attraction and in some instances can change the attraction to a weak repulsion, since it is now possible for the inner particle to centrifuge faster than the outer one. However the strongest effect of the lubrication force is restricted to particles within one or two radii of the wall and thus the overall effect on bulk properties is negligible for sufficiently large cylinders, at least at low particle concentrations. On the other hand the cylinder wall also screens the hydrodynamic interaction between particles that are far apart relative to the cylinder radius [6]. Wall screening is independent of the cylinder size and is therefore more important than the lubrication force.

Wall screening can be approximated to first order in the inverse separation by using image particles. The image of particle $i$ is located on the opposite side of the cylinder wall, $R_{\nu} = (2R - r_i, \theta_i, z_i)$ with instantaneous velocity $v_\nu = -v_i$. Based on the linearity of the Stokes equation, The relative axial velocity including wall screening, $w_{12,z}$, is therefore

$$w_{12,z} = v_{12,z} + \frac{3az}{4} \left[ u_c \left( \frac{r_{12} \cdot r_2}{R^{3/2}} - \frac{r_{12} \cdot r_1}{R^{3/2}} \right) - \frac{3az}{4} \left[ u_s \left( \frac{r_{12}^2}{R^{3/2}} - \frac{r_{12}^2}{R^{3/2}} \right) \right] \right]. \quad (4)$$

The additional terms on the right hand side describe the effects of screening on the centrifuging velocity and the settling velocity, respectively.

The effect of an axial drift velocity on the particle concentration $c(z, t)$ can be obtained from the mass balance,

$$\partial_t c(z, t) + \frac{3az}{4} \left[ u_c \left( \frac{r_{12} \cdot r_2}{R^{3/2}} - \frac{r_{12} \cdot r_1}{R^{3/2}} \right) - \frac{3az}{4} \left[ u_s \left( \frac{r_{12}^2}{R^{3/2}} - \frac{r_{12}^2}{R^{3/2}} \right) \right] \right] = -\frac{\partial_z}{\partial z} \int_{-\infty}^{\infty} c(z) \langle w_{12,z}(z - z') \rangle c(z', t) \pi R^2 dz', \quad (5)$$

where $\langle w_{12,z}(z) \rangle$ is the number of particles in an axial slice $dz$ located at $z'$. An infinite cylinder length is assumed so that end effects can be neglected, and $\langle w_{12,z}(z) \rangle$ is the spatial average over the $(r, \theta)$ plane of the relative velocity of pairs of particles separated axially by $z$;

$$\langle w_{12,z}(z) \rangle = \frac{2}{\pi R^4} \int_0^R \int_0^R \int_0^{2\pi} dr_1 dr_2 d\theta r_1 r_2 w_{12,z}(z), \quad (6)$$

where $\theta = \theta_2 - \theta_1$. Notice that $\langle w_{12,z}(z) \rangle$ is independent of the gravitational force, because the gravitational contribution to $w_{12,z}(z)$ (Eq. 4) averages to zero. A homogeneous distribution of particles in the $(r, \theta)$ plane has been assumed in the averaging, but the suspension is initially in a highly non-uniform state, with particles in a dense pack at the bottom of the cylinder. Even after some startup time the particles may not be distributed uniformly in the transversal plane, as assumed in Eq. 6. Nevertheless we do not expect our conclusions to be sensitive to radial variations in particle concentration, since all direct interactions, $v_{12,z}$, are attractive and long ranged.

We now consider the stability of a small sinusoidal perturbation to the concentration field, $c(z, t) = c_0 + c_k(t)e^{ikz}$. Linearizing Eq. 5 yields exponentially growing instabilities when

$$w_k = \int_{-\infty}^{\infty} dz e^{-ikz} \frac{d(w_{12,z}(z))}{dz} < 0. \quad (7)$$

The growth rate of each mode is given by

$$\gamma_k = -c_0 \pi R^2 w_k = \frac{\phi(\rho_s - \rho_f) \Omega^2 R^2}{8 \mu} f(k), \quad (8)$$

where $\phi$ is the particle volume fraction and $f(k)$ contains the wavelength dependence of the growth rate. The expression for $f(k)$ is a four dimensional integral over $r_1$, $r_2$, $\theta$, and $z$, and after integrating by parts,

$$f(k) = \int_{-\infty}^{\infty} dz k \sin(kz) \left\{ \frac{2 \pi r_1^2 R_1^2}{R_2^2} - 2 \left( \frac{2r_1 r_2}{R_2^2} \right) \right\}, \quad (9)$$

The integral over $z$ can be expressed in terms of the modified Bessel function $K_0$ [7], reducing the expression to spatial averages of the form given in Eq. 6. The Bessel functions $K_0(kr_{12})$ and $K_0(kR_1)$ can be expanded as products of functions of $r_1$, $r_2$, and $\theta$ [8] and the resulting integrals written in terms of modified Bessel functions of the 1st and 2nd kind [7]:

$$f_0 = 16 \left[ 2K_2 I_2 - K_0 I_2 - K_1 I_1 \right], \quad (10)$$

and

$$f_1(k) = 16I_0(k) \left[ K_2(k) + 4k^{-1} (K_i(k) - K_i(2k)) \right] - 16I_1(k) \left[ 8K_1(2k) - 8K_3(2k) + K_3(k) \right] - 64I_1(k) k^{-1} \left[ 4K_0(2k) - K_0(k) + 3k^{-1} K_i(k) \right], \quad (11)$$

FIG. 3: $f(k)$ is plotted against $kR$. The dotted line in the capped figure is Eq. 10.
We were surprised to discover that the initial segregation process is driven solely by differential centrifuging, although gravity may play a role in the later stages of band formation, when there are substantial inhomogeneities in particle concentration. In Fig. 4, the band onset time observed experimentally is compared with the mode growth time \( \gamma_k^{-1} \) of the most unstable mode. The band onset time was determined by measuring the time delay between cylinder startup and the first appearance of bands in the concentration profile. Our theory predicts that the band growth rate is proportional of \( \Omega^2/\mu \) (Eq. 8). Although there is large scatter in the experimental data, the band onset time is roughly proportional to \( \mu \Omega^{-2} \) over a range of viscosities (10–70 cp) and rotation periods (1–10 Hz). Moreover, with the exception of 2 outlying points, the experimental data is consistent with the predicted values of \( \gamma_k \), but a quantitative comparison awaits more precise measurements of the growth rate.

In this work, we have proposed a theory for axial segregation of a non-neutrally buoyant particle suspension in a rotating cylinder. The main idea is that the differential centrifuging at different radial positions drives an axial attraction between pairs of particles, which amplifies any existing concentration fluctuation. Our theory is consistent with the band width and the band onset time observed experimentally, and has no adjustable parameters. We note that in a suspension of particles that are less dense than the suspending medium, any axial perturbation in concentration will decay due to stabilizing repulsive interactions. In a suspension of neutrally buoyant particles, centrifuging is absent and the suspension is stable in the absence of a free surface [2].

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